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Symmetrized powers of point group representations

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Abstract. The problem of decomposing the symmetrized powers of representations of point groups is discussed in detail. Advantage is taken of the fact that each point group representation is closely related either to an induced linear character or to a representation of SU(2). The theory introduced by Gard in 1973 is developed to cover the symmetrization of induced linear characters and also representations induced from a normal subgroup.

1. Introduction

In the group-theoretical approach to quantum-mechanical many-body problems the bulk of the mathematical effort lies in the identification of the irreducible constituents of the Kronecker products of symmetry group representations. In particular, for a system of identical bodies, attention is focused on the reduction of the Kronecker powers of a single representation. Indeed, whenever the Pauli principle, or one of its generalizations is in operation, knowledge is required of the symmetrized powers of group representations carried by various natural subspaces of the tensor power spaces, called symmetry classes, which additionally carry primary representations of the symmetric group which permutes the sub-system labels. For a simple introduction to these ideas and a short list of applications see Bradley and Cracknell (1972) under the index entry 'symmetrized squares . . . '.

The mathematical problem of calculating the symmetrized powers of group representations, which in the special case of point groups is the subject of this paper, has been considered by various authors. In particular, Lewis (1973) and Boyle (1972) have made use of the character formula of Weyl (see Weyl 1950, p 331 or Lyubarskii 1960, p 75) to show how the totally symmetrized and totally antisymmetrized powers of space group and point group representations, respectively, may be computed. Unfortunately these must be considered special cases, because, although the Weyl formula may in principle be applied wherever characters make sense, its use rarely leads directly to useful formulae for the decomposition of a symmetrized power as a direct sum of irreducibles.

An alternative approach, specifically designed for induced representations, was initiated by Mackey (1953) in a paper which dealt with the problem of expressing the symmetrized square of an induced representation as a direct sum of induced representations. This method was taken up by Bradley and Davies (1970), who explored in more detail the computational aspects of its matrix formulation and its application to space groups. It was further exploited by Backhouse (1973) in a fresh look at the Herring test for the reality classification of space group representations. More recently, Mackey's method has been fully generalized by Gard (1973a), in the spirit of the Bradley and Davies formulation, so that an arbitrary symmetrized power of a finite-dimensional induced representation can be written as the direct sum of induced representations. This work was analysed more fully for space groups in Gard (1973b). In its general presentation this method is rather lengthy, however considerable simplifications occur when the representations are induced from linear characters of a normal subgroup. Fortunately this is the case for many of the irreducible representations of the three-dimensional point groups.

When the induced representation approach is inappropriate, we find it is frequently the case that the Lie group SU(2) can be brought into play. This is so because some irreducible point group representations are the restriction of irreducible representations of SU(2), which may easily be symmetrized following Gard and Backhouse (1974).

In this paper we have aimed at giving formulae, or easily applied rules, for calculating the irreducible constituents of the symmetrized powers of all of the irreducible representations of each of the three-dimensional point groups, thus completing the tabulations begun by Boyle.

The presentation of the paper is as follows. In § 2 the irreducible point group representations, both single- and double-valued, are examined critically and classified according to which symmetrization procedure seems the most convenient. The categories are the following: (*a*) an induced linear character; (*b*) induced from a normal subgroup; (*c*) the restriction of an irreducible representation of SU(2); (*d*) the inner Kronecker product of a linear character with a representation of type (*a*), (*b*) or (*c*); (*e*) the outer Kronecker product of a representation of type (*a*), (*b*), (*c*) or (*d*) with a representation of the inversion group C_i. This listing is exhaustive, but the categories are not disjoint. Hence there is scope for choosing a preferred method in each case.

In § 3 procedures for dealing with cases (*a*), (*b*), (*c*), (*d*), (*e*) are described. In particular a special case of the theory of Gard (1973a) is developed to cover cases (*a*) and (*b*). This is especially simple, and most useful, when the subgroup concerned is normal and of prime index. Secondly, formulae are quoted for the symmetrization of the low-dimensional representations of SU(2) which are directly relevant to (*c*). Finally, two special cases of plethysm formulae are given which relate to cases (*d*) and (*e*).

The final section contains the results of applying these procedures to point groups. Since our emphasis is on general formulae, it will be appreciated that standard point group notation is rather ill-suited to our calculations in some cases. We have therefore adopted our own notation for the cyclic and dihedral groups. Otherwise the labelling will be as found in Bradley and Cracknell (1972). We have considered in detail only the proper rotation groups; those containing improper rotations may be tackled by taking advantage of the isomorphisms between them and direct products of those already discussed.

2. Irreducible representations

2.1. Cyclic groups

If m is finite, C_m is generated by a (ie c_{mz} in usual notation), where $a^m = e$, the identity. The double group $\bar{C}_m$ is isomorphic to C_{2m}, and has the relations $\bar{a}^m = r, r^2 = e$, where, in concrete terms, r is associated with the element diag$\{-1, -1\}$ of SU(2).

The unitary irreducible representations (UIR) of C'_m we denote by

$$\chi^l (l = 0, 1, \ldots, 2m-1),$$

where

$$\chi^l(\bar{a}^t) = \exp\left(\frac{2\pi i l t}{2m}\right). \tag{2.1}$$

Observe that χ^l gives rise to a single- or double-valued UIR of C_m according as l is even or odd.

If m is infinite, C_∞ can be identified with the circle group of complex numbers $e^{i\phi}$, $0 \leqslant \phi < 2\pi$. The double group C'_∞ is the multiplicative group of complex numbers $e^{i\phi/2}, 0 \leqslant \phi < 4\pi$. Thus the UIR of C'_∞ are $\chi^l(l = 0, \pm 1, \ldots)$ where

$$\chi^l(e^{i\phi/2}) = e^{il\phi/2}. \tag{2.2}$$

As above, χ^l gives rise to a single- or double-valued UIR of $C_\infty \cong C'_\infty/\{1, -1\}$ according as $|l|$ is even or odd.

2.2. *Dihedral groups*

The dihedral group D_m is the semi-direct product of the normal subgroup C_m with a 2-group $\{e, b\}$. It suffices to note that b acts as the inversion operation on C_m.

The double group D'_m has C'_m as a halving subgroup, and we may write

$$D'_m = C'_m \cup \bar{b}C'_m;$$

however, D'_m is not isomorphic to D_{2m} since the set $\{e, \bar{b}\}$ does not form a subgroup of order two. Although $\bar{b}$ acts as the inversion operation on C'_m, we find $\bar{b}^2 = r$, where r is the element mentioned in § 2.1. In concrete terms, $\bar{b}$ can be interpreted as the element $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ of SU(2), though this is not the standard identification in all cases.

The little group procedure, based on the subgroup C'_m, yields the following UIR:

$$\phi^l = \chi^l \uparrow D'_m \equiv \chi^{2m-l} \uparrow D'_m, \tag{2.3}$$

for finite m and $l = 1, 2, \ldots, m-1$, where χ^l is defined by (2.1). In addition to these two-dimensional UIR, there are four linear characters which are extensions of χ^0 and χ^m. They are defined by

$$\begin{aligned} \phi_1^0(\bar{a}) &= \phi_2^0(\bar{a}) = \chi^0(\bar{a}) = 1 \\ \phi_1^0(\bar{b}) &= -\phi_2^0(\bar{b}) = 1 \end{aligned} \tag{2.4}$$

$$\begin{aligned} \phi_1^m(\bar{a}) &= \phi_2^m(\bar{a}) = \chi^m(\bar{a}) = -1 \\ \phi_1^m(\bar{b}) &= -\phi_2^m(\bar{b}) = \begin{cases} 1 & \text{if } m \text{ is even} \\ i & \text{if } m \text{ is odd.} \end{cases} \end{aligned} \tag{2.5}$$

Evidently ϕ^l, for l odd, and ϕ_1^m, ϕ_2^m, m odd, are double-valued and the remainder are single-valued on D_m.

If m is infinite, (2.3) and (2.4) are valid, provided χ^l is defined by (2.2), χ^{2m-l} is interpreted as χ^{-l}, and $\bar{a}$ is replaced by $e^{i\phi/2}$ in (2.4). Equations (2.5) are suppressed.

We remark finally that ϕ^1 is the restriction to D'_m of the UIR $D^{1/2}$ of SU(2). Unfortunately, not all the double-valued representations of D_m are related simply to ϕ^1.

2.3. *Tetrahedral group* $\boldsymbol{T}$

In standard notation, the four single-valued UIR of $\boldsymbol{T}$ are denoted by A, ${}^1E$, ${}^2E$, T. The first three are linear characters, and the fourth is the restriction to $\boldsymbol{T}$ of the three-dimensional representation D^1 of SO(3). Alternatively, T may be induced from any of the non-trivial characters, B_1, B_2, B_3 of the normal subgroup $\boldsymbol{D}_2$.

$\boldsymbol{T}$ has three two-dimensional double-valued UIR, denoted by $\bar{E}$, ${}^1\bar{F}$, ${}^2\bar{F}$. We find that $\bar{E} = D^{1/2} \downarrow \boldsymbol{T}'$, ${}^1\bar{F} = \bar{E} \otimes {}^2E$, ${}^2\bar{F} = \bar{E} \otimes {}^1E$.

2.4. *Octahedral group* $\boldsymbol{O}$

$\boldsymbol{O}$ has two linear characters denoted by A_1 and A_2. The single-valued two-dimensional UIR, denoted by E, may be induced from either of the linear characters ${}^1E$ and ${}^2E$ of the normal subgroup $\boldsymbol{T}$. The remaining single-valued UIR, T_1 and T_2, are three-dimensional and may be constructed in two different ways: either as the induced representations $T_1 = A_2 \uparrow \boldsymbol{O}$ and $T_2 = B_2 \uparrow \boldsymbol{O}$, where A_2, B_2 are linear characters of the non-normal subgroup $\boldsymbol{D}_4$, or by noting that $T_1 = D^1 \downarrow \boldsymbol{O}$ and $T_2 = A_2 \otimes T_1$.

The double-valued UIR of $\boldsymbol{O}$ are most easily related to the UIR $D^{1/2}$ and $D^{3/2}$ of SU(2). In fact $\bar{E}_1 = D^{1/2} \downarrow \boldsymbol{O}'$, $\bar{E}_2 = \bar{E}_1 \otimes A_2$ and $\bar{F} = D^{3/2} \downarrow \boldsymbol{O}'$. Alternatively, $\bar{F}$ can be written as ${}^1\bar{E} \uparrow \boldsymbol{O}'$, ${}^2\bar{E} \uparrow \boldsymbol{O}'$, ${}^1\bar{F} \uparrow \boldsymbol{O}'$, ${}^2\bar{F} \uparrow \boldsymbol{O}'$, where ${}^1\bar{E}$, ${}^2\bar{E}$ are the double-valued linear characters of the non-normal subgroup $\boldsymbol{D}_3'$, and ${}^1\bar{F}$, ${}^2\bar{F}$ are double-valued two-dimensional UIR of the normal subgroup $\boldsymbol{T}'$.

3. Symmetrization procedures

3.1. *Symmetrized powers of induced linear characters*

In a recent paper, Gard (1973a, to be referred to as I), a procedure was derived for the construction of the symmetrized powers of a finite-dimensional induced representation. As a preliminary to discussing what simplifications may occur in the general theory, let us recall some of the notation and results of I.

Let Δ be a representation of a group $\boldsymbol{G}$ with basis $\{\psi_1, \psi_2, \ldots, \psi_f\}$. The vector space Ω spanned by ordered n-tuples of functions $(\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_n})$, $i_s = 1, 2, \ldots, f$; $s = 1, 2, \ldots, n$, is the carrier space for the nth Kronecker power of Δ. There is an action of the permutation group $\boldsymbol{S}_n$ on the positions of the functions in the n-tuple which commutes with the action of $\boldsymbol{G}$. Hence Ω splits into a direct sum of $\boldsymbol{G} \times \boldsymbol{S}_n$-invariant subspaces Ω^ν in one–one correspondence with the UIR of $\boldsymbol{S}_n$. Ω^ν has the important property that it carries a representation $\langle\nu\rangle \otimes [\nu]$ of $\boldsymbol{G} \times \boldsymbol{S}_n$, where $[\nu]$ is a UIR of $\boldsymbol{S}_n$ of dimension f_ν and $\langle\nu\rangle$ is some UR of $\boldsymbol{G}$ of dimension d_ν. This with the proviso that $\Omega^\nu = \{0\}$ if dim Δ is less than the number of rows in the Young's diagram of $[\nu]$.

Now let D be a UR of the subgroup $\boldsymbol{K}$ of $\boldsymbol{G}$ with basis $\{\phi_1, \phi_2, \ldots, \phi_d\}$. Define inductively the double coset decompositions

$$\boldsymbol{G} = \bigcup_j \boldsymbol{K}_{r-1} d_j^r \boldsymbol{K}, \tag{3.1}$$

$r = 0, 1, \ldots, n-1$, where $\boldsymbol{K}_{-1} = \boldsymbol{G}$, $\boldsymbol{K}_0 = \boldsymbol{K}$ and

$$\boldsymbol{K}_r = \boldsymbol{K} \cap \bigcap_{i=1}^{r} d_{\alpha_i} \boldsymbol{K} d_{\alpha_i}^{-1}, \tag{3.2}$$

$r = 1, \ldots, n-1$. Since $\boldsymbol{K}_r$ depends on the fixed set $d_{\alpha_i} \in \{d_j^i\}$, $i = 1, \ldots, r$, of double coset representatives chosen, it is sometimes convenient to give $\boldsymbol{K}_r$ the superscript (α) which denotes the n-tuple $(d_{\alpha_{n-1}}, \ldots, d_{\alpha_1}, d_{\alpha_0} = e)$. Now write

$$\boldsymbol{K}_{r-1}^{(\alpha)} = \bigcup_{\sigma} q_\sigma^r \boldsymbol{K}_r^{(\alpha)}, \tag{3.3}$$

for $r = 0, 1, \ldots, n-1$, where we have suppressed the dependence of the q on (α). Then the nth Kronecker power of $D \uparrow \boldsymbol{G}$ may be expressed as

$$(D \uparrow \boldsymbol{G})^n \equiv \bigoplus_{(\alpha)} [(D_{\alpha_{n-1}} \downarrow \boldsymbol{K}_{n-1}) \otimes \ldots \otimes (D_{\alpha_0} \downarrow \boldsymbol{K}_{n-1})] \uparrow \boldsymbol{G} \tag{3.4}$$

where $\boldsymbol{K}_{n-1} = \boldsymbol{K}_{n-1}^{(\alpha)}$, the direct sum ranges over all chains of double coset representatives (α) and D_α is the UR of $d_\alpha \boldsymbol{K} d_\alpha^{-1}$ defined by

$$D_\alpha(d_\alpha k d_\alpha^{-1}) = D(k), \tag{3.5}$$

for all $k \in \boldsymbol{K}$. This is associated with a decomposition of the space Ω as $\oplus_{(\alpha)}\Omega(\alpha)$. That is a direct sum decomposition of $\boldsymbol{G}$-invariant subspaces

$$\Omega(\alpha) = \sum q_\sigma^0 q_{\sigma'}^1 \ldots q_{\sigma''}^{n-1}(d_{\alpha_{n-1}}\phi_{i_{n-1}}, \ldots, d_{\alpha_0}\phi_{i_0}), \tag{3.6}$$

where, for fixed (α), the sum is over all $\sigma, \sigma', \ldots$; $i_s = 1, 2, \ldots, d$; $s = 0, 1, \ldots, n-1$. Also the summation is taken to mean the linear span of all the functions appearing to the right of it.

In § 3 of I, an action of $\boldsymbol{S}_n$ was defined which permuted the n-tuples (α) and hence the subspaces $\Omega(\alpha)$, causing aggregates of them to form orbits. This naturally led to the subgroups $\boldsymbol{S}_n(\alpha)$ and $\boldsymbol{E}_n(\alpha)$, the former being the stability group of $\Omega(\alpha)$ and the latter the stability group of (α) (see equations (3.1)–(3.6) of I). The direct sum of the spaces belonging to the orbit of $\Omega(\alpha)$, called $T(\alpha)$, is $\boldsymbol{G} \times \boldsymbol{S}_n$-invariant by construction. It is the spaces $T(\alpha)$, rather than the $\Omega(\alpha)$, which split conveniently into subspaces which may be allocated to the appropriate symmetry classes. The procedure for obtaining $T(\alpha)$ is to induce from $\boldsymbol{K}_{n-1}^{(\alpha)}$ up to an intermediate subgroup $\boldsymbol{M}(\alpha)$—which may be described roughly as $\boldsymbol{K}_{n-1}^{(\alpha)}$ augmented by those permutations of $\boldsymbol{S}_n(\alpha)$ not present in $\boldsymbol{E}_n(\alpha)$. The intermediate space, denoted by $W(\alpha)$ in I, is not only invariant under $\boldsymbol{M}(\alpha)$ but also under $\boldsymbol{S}_n$. The representation carried by $W(\alpha)$ can be split up explicitly so that the resulting subrepresentations may be allocated to subspaces, which on further induction to $\boldsymbol{G}$, remain wholly within their predetermined Ω^ν. In I a construction was given for the group $\boldsymbol{M}(\alpha)$ which followed the discovery of the group $\boldsymbol{S}_n(\alpha)$. In particular, if $\boldsymbol{S}_n(\alpha) = \boldsymbol{E}_n(\alpha)$, it was found that $\boldsymbol{M}(\alpha) = \boldsymbol{K}_{n-1}^{(\alpha)}$, and the resulting representations, with their allocation to symmetry classes were given in equation (6.28). If $\boldsymbol{S}_n(\alpha)$ is strictly greater than $\boldsymbol{E}_n(\alpha)$, the construction of $\boldsymbol{M}(\alpha)$ turns on the conclusions of theorem (3.7), and the results concerning these representations are contained in equation (6.15), for which we take this opportunity of noting the following correction. If $\sigma = (\sigma_1, \ldots, \sigma_r) \in \boldsymbol{E}_n(\alpha)$ is regarded as an element of $\boldsymbol{S}_r$, then in (6.12) $(\sigma^{p_t - p_{t-1}})$ should be replaced by $(\sigma_t \sigma_{\pi^{-1}(t)} \ldots \sigma_{\pi^{-q}(t)})$, where $(q+1)$ is the length of the cycle of π containing t, with a corresponding change in (6.14). More detailed analysis has revealed the following theorem, which enables $\boldsymbol{M}(\alpha)$ and $\boldsymbol{S}_n(\alpha)$ to be determined simultaneously.

Theorem (3.1).
Let $\pi \in S_n(\alpha)$, then
(i)

$$d_{\alpha_r} = p_{n-1} \dots p_0 d^{-1}_{\alpha_{\pi(0)}} d_{\alpha_{\pi(r)}} k_{r-1}, \tag{3.7}$$

$r = 1, \dots, n-1$, where $p_i \in K_i^{(\alpha)}$, $k_i \in K$, $i = 0, \dots, n-2$;
(ii) it is possible to choose

$$a_\pi = d_{\alpha_{\pi(0)}} p_0^{-1} \dots p_{n-1}^{-1}; \tag{3.8}$$

(iii) with this choice of a_π, the matrix P has components given by

$$P_{ij} = \prod_{s=1}^{r} \langle \mu_s \rangle (k_{s-1})_{i_s j_s}. \tag{3.9}$$

Proof.
(i) By (2.1) of I, $(\alpha) \sim (\beta)$ if

$$d_{\alpha_r} = p_{r-1} \dots p_0 d_{\beta_r} k'_{r-1}, \tag{3.10}$$

$r = 1, \dots, n-1$, where $p_i \in K_i^{(\alpha)}$, $k'_i \in K$, $i = 0, \dots, n-2$. But if $i \geqslant r$, $p_i \in K_i^{(\alpha)} \leqslant d_{\alpha_r} K d^{-1}_{\alpha_r}$, hence (3.10) may be rewritten as

$$d_{\alpha_r} = p_{n-1} \dots p_0 d_{\beta_r} k_{r-1}, \tag{3.11}$$

$r = 1, \dots, n-1$, where $k_i \in K$, $i = 0, \dots, n-2$. Finally, (3.7) holds since $\pi \in S_n(\alpha)$ means that (α) and $\hat{\pi}(\alpha)$ are equivalent (see § 3 of I).
(ii) It follows from (i) that $d_{\alpha_{\pi(0)}} p_0^{-1} \dots p_{n-1}^{-1} \in d_{\alpha_{\pi(r)}} K d^{-1}_{\alpha_r}$ for all r. But by theorem (3.6) of I,

$$a_\pi \in \bigcap_{s=0}^{n-1} d_{\alpha_{\pi(s)}} K d^{-1}_{\alpha_s},$$

hence result.
(iii) Substituting (3.8) in equation (6.3) of I, and using (3.7), leads to (3.9).

Corollary (3.2).
If $K \lhd G$, then a_π may be chosen to be $d_{\alpha_{\pi(0)}}$.

Proof.
The normality of K in G implies each $K_r^{(\alpha)} = K, r = 1, \dots, n-1$. Double coset representatives become left coset representatives for K in G and hence we may choose $p_i = e$, for $i = 0, \dots, n-1$.

Corollary (3.3).
If $G = K \circledS B$, with $K \lhd G$, and the double coset representatives are chosen to lie in B, then $P = I$.

Proof.
If $\pi \in S_n(\alpha)$ then $d^{-1}_{\alpha_{\pi(0)}} d_{\alpha_{\pi(s)}} \in d_{\alpha_s} K$, for $s = 0, 1, \dots, n-1$ follows from (3.7). But B, which consists of double coset representatives, is closed under multiplication, and hence $d^{-1}_{\alpha_{\pi(0)}} d_{\alpha_{\pi(s)}} = d_{\alpha_s}$, for all s. We have already chosen $p_i = 0$, $i = 0, \dots, n-1$, so now we find $k_s = e$, $s = 0, \dots, n-2$, with the result that $P = I$. This concludes the proof.

In the sequel take $K \lhd G$, and write $G/K = B$. As stated above, double coset representatives are left coset representatives for K in G but they are not necessarily closed under multiplication. Let $p: G \to B$ be the canonical epimorphism. With each n-tuple (α) of elements of G associate an n-tuple $(\bar{\alpha})$ of elements of B defined by

$$(\bar{\alpha}) = (p(d_{\alpha_{n-1}}), \ldots, p(d_{\alpha_0})). \tag{3.12}$$

Then we have the following theorem.

Theorem (3.4).
Let $K \lhd G, G/K = B$ and let $\{\bar{\alpha}\}$ denote the set of entries in the n-tuple $(\bar{\alpha})$. If $B(\alpha)$ is the largest subgroup of B contained in $\{\bar{\alpha}\}$ with the property that $\{\bar{\alpha}\}$ consists of whole cosets of $B(\alpha)$, then $M(\alpha) = p^{-1}B(\alpha)$.

Proof.
First note that by corollary (3.2), K and a subset of $\{\alpha\}$ generate $M(\alpha)$.

Let $(\alpha) = (g_t, \ldots, g_t, \ldots, g_1, \ldots, g_1, g_0, \ldots, g_0)$ where $g_0 = e$ and g_i appears r_i times, $i = 0, \ldots, t$. Let $\pi \in S_n(\alpha)$, then we may take $a_\pi = d_{\alpha_{\pi(0)}} = g_i$, say. $\hat{\pi}(\alpha) \sim \alpha$ implies that given j, there exists k, dependent on $j, k(j) \neq k(j')$ if $j \neq j'$, so that $g_i^{-1} g_k \in g_j K$ and $r_j = r_k$. Equivalently we may write

$$p(g_k) = p(g_i)p(g_j), \tag{3.13}$$

with $r_j = r_k$. Each g_i, which arises in this way from some $\pi \in S_n(\alpha)$, provides a one–one correspondence between the distinct elements of $\{\bar{\alpha}\}$. Conversely if $g_i \in \{\alpha\}$ satisfies (3.13), then it defines $\pi \in S_n(\alpha)$ with $a_\pi = g_i$. The totality of the distinct $p(g_i)$ satisfying (3.13) forms a subgroup $B(\alpha)$ of B and $pM(\alpha) = B(\alpha)$. We must now show that $B(\alpha)$ decomposes $\{\bar{\alpha}\}$ and is maximal with respect to this property.

If $g_s \in \{\alpha\}$, then the set $A_s = \{bp(g_s) : b \in B(\alpha)\}$ consists of distinct elements of $\{\bar{\alpha}\}$, since $K \lhd G$, and by (3.13) they appear exactly r_s times in $\{\bar{\alpha}\}$. Either the elements of $B(\alpha)$ and A_s, with their multiplicities, fill $\{\bar{\alpha}\}$, or there exists $g_{s'} \in \{\alpha\}$ such that $p(g_{s'}) \notin B(\alpha) \cup A_s$. It is trivial to show that $A_{s'} \cap (B(\alpha) \cup A_s)$ is empty. Hence $\{\bar{\alpha}\}$ is the union of left cosets of $B(\alpha)$ in B.

Finally, suppose $\{\bar{\alpha}\}$ may be written as the union of left cosets of a subgroup B' of B, where $B(\alpha)$ is contained in B'. Clearly any $b \in B'$ satisfies (3.13) and we can define $\pi \in S_n(\alpha)$ such that $a_\pi = b$. We deduce that B' is contained in $B(\alpha)$, hence $B(\alpha)$ is maximal.

Corollary (3.5).
If B is cyclic of prime order, then either $B(\alpha) = B$ or $B(\alpha) = \{e\}$. In the first case $\{\bar{\alpha}\} = mB$ for some integer m.

Corollary (3.6).
Let $G = K \circledS B$, then there exists for each (α) a maximal subgroup $\bar{B}(\alpha)$ of B such that $\{\alpha\}$ consists entirely of left cosets of $\bar{B}(\alpha)$, and $M(\alpha) = K \circledS \bar{B}(\alpha)$.

We have described how to generate the group $M(\alpha)$, generally and in special cases, so it only remains to examine more closely the symmetrization procedure as it applies to induced linear characters. The first point to notice is that each pre-induced space, invariant under K_{n-1} and indexed by a double coset representative, carries a one-dimensional representation. Hence we may apply the theory following equation (6.19) of I. Let (α) be a standard n-tuple of type $(\lambda_1, \ldots, \lambda_r)$, leading of course to

$E_n(\alpha) = S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_r}$. Then there will be a contribution to the symmetry class Ω^ν only if $g_\nu \neq 0$ in

$$([\lambda_1] \otimes [\lambda_2] \otimes \ldots \otimes [\lambda_r]) \uparrow S_n = \bigoplus_\nu g_\nu[\nu], \tag{3.14}$$

where $[\lambda_i]$ denotes the trivial representation of S_{λ_i}, $i = 1, \ldots, r$. Now $\Omega(\alpha)$ carries a representation of G which we denote by $\phi_\alpha \uparrow G$, being the summand of (3.4) indexed by (α). Hence the representation $\Gamma^{(\alpha)}_{[\nu]}$ of $M(\alpha)$ (see (5.14) and (6.21) of I) is defined by

$$\Gamma^{(\alpha)}_{[\nu]}(l) = g_\nu f_\nu \phi_\alpha(l), \tag{3.15}$$

$$\Gamma^{(\alpha)}_{[\nu]}(a_\pi) = g_\nu[\nu](\pi) \otimes P(\pi), \tag{3.16}$$

where $l \in K^{(\alpha)}_{n-1}$ and $P(\pi)$ is a complex number. It will be recalled that Ω^ν carries the direct sum of the $\Gamma^{(\alpha)}_{[\nu]} \uparrow G$, one (α) from each orbit. In particular if $M(\alpha) = K^{(\alpha)}_{n-1}$, $T(\alpha)^\nu$ carries $f_\nu g_\nu(\phi_\alpha \uparrow G)$.

Now consider the case when $G/K = B$ is a cyclic group of prime order q, and denote the coset representatives of K in G by $e, b, b^2, \ldots, b^{q-1}$. Suppose the n-tuple (α) contains b^i, λ_i times, $i = 0, \ldots, q-1$, $\lambda_0 \neq 0$. By corollary (3.5), if at least one $\lambda_i \neq n/q$, then $S_n(\alpha) = E_n(\alpha) = S_{\lambda_0} \times S_{\lambda_1} \times \ldots \times S_{\lambda_{q-1}}$. Then $M(\alpha) = K^{(\alpha)}_{n-1} = K$, and $T(\alpha)^\nu$ carries the UR $g_\nu f_\nu(\phi_\alpha \uparrow G)$. The exceptional case occurs when q divides n and every λ_i is n/q. Evidently $b \in M(\alpha)$ and $M(\alpha) = G$. The decomposition is as follows. Possible $[\nu]$ are those for which $g_\nu \neq 0$ in

$$([n/q] \otimes [n/q] \otimes \ldots \otimes [n/q]) \uparrow S_n = \bigoplus_\nu g_\nu[\nu]. \tag{3.17}$$

If $k \in K$, then $T(\alpha)^\nu$ carries $\Gamma^{(\alpha)}_{[\nu]}$ defined by

$$\Gamma^{(\alpha)}_{[\nu]}(k) = g_\nu f_\nu \phi_\alpha(k), \tag{3.18}$$

$$\Gamma^{(\alpha)}_{[\nu]}(b) = g_\nu[\nu](\pi_b)P(\pi_b), \tag{3.19}$$

where π_b is in the class $[q^{n/q}]$ of S_n. Note that $P(\pi_b) = 1$ if $G = K \circledS B$, by theorem (3.1) (iii).

The permutation π_b has prime order q, and hence the eigenvalues of $[\nu](\pi_b)$ are qth roots of unity. If we put $\omega_t = \exp(2\pi i t/q)$, $t = 0, 1, \ldots, q-1$, and if $[\nu](\pi_b)$ has the eigenvalue ω_t with multiplicity $a^{(\nu)}_t$, then we know the diagonal form of $[\nu](\pi^r_b)$ for $r = 0, 1, \ldots, q-1$. Indeed, since π_b is a product of disjoint q-cycles, q a prime, all the non-trivial powers of π_b have the same cycle structure and hence the same character in the representation $[\nu]$. A consideration of the diagonal form of $[\nu](\pi_b)$ and its powers, together with the invariance of the character function and the irreducibility of the polynomial $1+x+x^2+\ldots+x^{q-1}$, leads to the conclusion that $a^{(\nu)}_1 = a^{(\nu)}_2 = \ldots = a^{(\nu)}_{q-1}$. Hence

$$f_\nu = a^{(\nu)}_0 + (q-1)a^{(\nu)}_1, \tag{3.20}$$

$$\mathrm{Tr}[\nu](\pi_b) = a^{(\nu)}_0 - a^{(\nu)}_1 = \frac{1}{q-1}(qa^{(\nu)}_0 - f_\nu). \tag{3.21}$$

In any particular case we may obtain f_ν and $\mathrm{Tr}[\nu](\pi_b)$ from the character table of S_n, thus enabling $[\nu](\pi_b)$ to be written in diagonal form and the representation $\Gamma^{(\alpha)}_{[\nu]}$ to be reduced to a direct sum of one-dimensional UR of G. For example if $q = 2$ and n is even, then $[n](\pi_b) = 1$ and $[1^n](\pi_b) = (-1)^{n/2}$. If q is odd and q divides n, then π_b is an even permutation and $[n](\pi_b) = [1^n](\pi_b) = 1$.

3.2. *Symmetrized powers of internal and external Kronecker products of a representation with a linear character*

Let G be a direct product group, $G = G_1 \times G_2$, and let D be a UR of G with the special form $D = D_1 \otimes \chi$, where D_1 is a UR of G_1 and χ is a linear character of G_2. Then

$$D^{[\nu]} = D_1^{[\nu]} \otimes \chi^n. \tag{3.22}$$

In particular if $G_1 = G_2 = H$, and we restrict to the diagonal subgroup, then (3.22) gives a formula for the symmetrized powers of the inner product of a UR of H with a linear character of H.

3.3. *Symmetrized powers of SU(2) representations*

Let Γ be a representation of the point group G given by $\Gamma = D^j \downarrow G$, where D^j is a UIR of SU(2) (or SO(3)). Then $\Gamma^{[\nu]} = (D^j)^{[\nu]} \downarrow G$. Following the recent paper (Gard and Backhouse 1974), we may decompose $(D^j)^{[\nu]}$ into its irreducible constituents. This together with the tables (2.7) and (6.6) of Bradley and Cracknell (1972), enables the reduction of $\Gamma^{[\nu]}$ to be completed. We highlight some useful results from Gard and Backhouse (1974).

Let $[\nu] = (\nu_1, \nu_2, \ldots, \nu_n)$ where $\nu_1 \geqslant \nu_2 \geqslant \ldots \geqslant \nu_n$ and $\Sigma_{i=1}^n \nu_i = n$. Then

$$(D^{1/2})^{[\nu]} = \begin{cases} D^{\frac{1}{2}(\nu_1 - \nu_2)} & \text{if } \nu_3 = 0, \\ \text{empty}, & \text{otherwise.} \end{cases} \tag{3.23}$$

Define $D^{[n]} = (D^1)^{[n]}$, then

$$D^{[n]} = D^n + D^{n-2} + \ldots + \begin{cases} D^0, & n \text{ even} \\ D^1, & n \text{ odd.} \end{cases} \tag{3.24}$$

Also

$$(D^1)^{[\nu]} = D^{[\nu_1-\nu_3]} + D^{[\nu_1-\nu_3-1]} + \ldots + D^{[\nu_2-\nu_3]} - (D^{[\nu_1-\nu_2-1]} + \ldots + D^{[0]}), \tag{3.25}$$

if $\nu_4 = 0$, but if $\nu_4 \neq 0$ there is no contribution to $(D^1)^{[\nu]}$.

4. Symmetrized powers of point group representations

In this section we describe the calculations required to obtain the symmetrized powers of the UIR of the point groups. It has not been possible to tabulate fully the irreducible constituents of every symmetrized power, because the formulae we give often depend on such parameters as $f_\nu, g_\nu, a_0^{(\nu)}, a_1^{(\nu)}$ which must be calculated in each individual case.

4.1. *Cyclic groups*

Take C_m' as the source for both the single- and double-valued UIR of C_m. Then, since the UIR are one dimensional, there is only a contribution to the totally symmetrized nth power, for each n, given by

$$(\chi^l)^{[n]} = \chi^{nl}, \tag{4.1}$$

where nl must be reduced modulo $2m$ if m is finite.

4.2. Dihedral groups

Take D'_m as the source for both single- and double-valued UIR of D_m. The one-dimensional representations, (2.4) and (2.5) if m is finite, but just (2.4) if m is infinite, contribute to the totally symmetrized powers only. The results are as follows

$$(\phi_1^0)^{[n]} = \phi_1^0; \tag{4.2}$$

$$(\phi_2^0)^{[n]} = \begin{cases} \phi_1^0, & n \text{ even} \\ \phi_2^0, & n \text{ odd.} \end{cases} \tag{4.3}$$

m even

$$(\phi_1^m)^{[n]} = (\phi_2^m)^{[n]} = \phi_1^0, \qquad n \text{ even}; \tag{4.4}$$

$$(\phi_j^m)^{[n]} = \phi_j^m, \qquad n \text{ odd}, \tag{4.5}$$

$j = 1, 2.$

m odd

If $n = 2r$

$$(\phi_j^m)^{[n]} = \begin{cases} \phi_1^0, & r \text{ even} \\ \phi_2^0, & r \text{ odd.} \end{cases} \tag{4.6}$$

If $n = 2r+1$

$$(\phi_1^m)^{[n]} = \begin{cases} \phi_1^m, & r \text{ even} \\ \phi_2^m, & r \text{ odd,} \end{cases} \tag{4.7}$$

$$(\phi_2^m)^{[n]} = \begin{cases} \phi_2^m, & r \text{ even} \\ \phi_1^m, & r \text{ odd.} \end{cases} \tag{4.8}$$

The two-dimensional UIR are given by (2.3) and the symmetrization procedure is based on theorem (3.1) (iii), corollary (3.2), theorem (3.4), corollary (3.5) and equations (3.17)–(3.21). We take $G = D'_m$, $K = C'_m$ and $G/K \simeq C_2$, which is cyclic of prime order.

We take the elements of the n-tuple (α) to be e, u times, where $u \neq 0$, and $\bar{b}$, v times.

If $u \neq v$, the space $T(\alpha)^{[v]}$ carries the UR $g_v f_v(\phi_\alpha \uparrow D'_m)$ where $\phi_\alpha = \chi^{(u-v)l}$ is a UIR of C'_m. Here g_v is the frequency of $[v]$ in $([u] \otimes [v]) \uparrow S_n$, the inducing subgroup being $S_u \times S_v$, $u+v = n$.

The exceptional case is where n is even, $u = v = n/2$, so that $\phi_\alpha = \chi^0$ and $M(\alpha) = D'_m$. Then $T(\alpha)^{[v]}$ carries the UR $\Gamma_{[v]}$ defined by

$$\Gamma_{[v]}(\bar{a}) = g_v f_v I, \tag{4.9}$$

$$\Gamma_{[v]}(\bar{b}) = g_v [v](\pi_{\bar{b}}) \otimes \chi^{ln/2}(r), \tag{4.10}$$

where $\bar{a} \in C'_m$, $\pi_{\bar{b}}$ is in the class $(2^{n/2})$. Note that $\chi^{ln/2}(r) = \exp(\pi i l n/2)$. Using (3.20), (3.21) we may decompose $\Gamma_{[v]}$ as a sum of $g_v a_0^{[v]}$ UIR ϕ_1^0, and $g_v a_1^{[v]}$ UIR ϕ_2^0 if l is even or if l is odd and $n/2$ is even. The frequencies are reversed if l is odd and $n/2$ is odd. In particular if $[v] = [n]$, $a_0^{[v]} = 1$, $a_1^{[v]} = 0$, but if $[v] = [1^n]$, $a_0^{[v]} = 1$, $a_1^{[v]} = 0$ if $n/2$ is even and vice versa if $n/2$ is odd.

The above is valid when m is infinite with suitable interpretation.

4.3. Tetrahedral group

The one-dimensional representations of $\boldsymbol{T}$ only contribute to the totally symmetrized nth power.

$$A^{[n]} = A; \tag{4.11}$$

$$({}^1E)^{[n]} = \begin{cases} A & \text{if } 3/n, \\ {}^1E & \text{if } 3/n-1, \\ {}^2E & \text{if } 3/n-2; \end{cases} \tag{4.12}$$

$$({}^2E)^{[n]} = \begin{cases} A & \text{if } 3/n, \\ {}^1E & \text{if } 3/n-2, \\ {}^2E & \text{if } 3/n-1. \end{cases} \tag{4.13}$$

Since $T = D^1 \downarrow \boldsymbol{T}$ we may use the theory of § 3.3 to decompose $(T)^{[\nu]}$. This approach is also applicable to the double-valued UIR, so from $\bar{E} = D^{1/2} \downarrow \boldsymbol{T}'$, we obtain

$$(\bar{E})^{[\nu]} = D^{\frac{1}{2}(\nu_1 - \nu_2)} \downarrow \boldsymbol{T}', \tag{4.14}$$

if $\nu_3 = 0$, and is empty otherwise. From (3.22) we find

$$({}^1\bar{F})^{[\nu]} = (\bar{E})^{[\nu]} \otimes ({}^2E)^{[n]}, \tag{4.15}$$

$$({}^2\bar{F})^{[\nu]} = (\bar{E})^{[\nu]} \otimes ({}^1E)^{[n]}, \tag{4.16}$$

which may be reduced with the aid of (4.12)–(4.14).

4.4. Octahedral group

Symmetrizing the linear characters gives

$$(A^1)^{[n]} = A^1, \tag{4.17}$$

$$(A^2)^{[n]} = \begin{cases} A^1, & n \text{ even} \\ A^2, & n \text{ odd.} \end{cases} \tag{4.18}$$

The two-dimensional single-valued UIR of $\boldsymbol{O}$ may be expressed as $E = {}^1E \uparrow \boldsymbol{O}$ where ${}^1E$ is a linear character of $\boldsymbol{T}$. Since $\boldsymbol{O} = \boldsymbol{T} \circledS \boldsymbol{C}_2$ we may apply the strongest form of the theory of §(3.1). Let $\boldsymbol{C}_2 = \{e, c\}$, then we may assume that an n-tuple (α) contains e, u times, $u \neq 0$, and c, v times, $u+v = n$. If $u \neq v$, $T(\alpha)^{[\nu]}$ carries the UR $g_\nu f_\nu(\phi_\alpha \uparrow \boldsymbol{O})$, where $\phi_\alpha = ({}^1E)^u({}^2E)^v$. If n is even and $u = v = n/2$, then $\phi_\alpha = A$, $\boldsymbol{M}(\alpha) = \boldsymbol{O}$. Hence $\Gamma_{[\nu]}$ is given by

$$\begin{aligned} \Gamma_{[\nu]}(t) &= g_\nu f_\nu I, \\ \Gamma_{[\nu]}(c) &= g_\nu[\nu](\pi_c), \end{aligned} \tag{4.19}$$

where $t \in \boldsymbol{T}$ and π_c is in the class $(2^{n/2})$. The matrix $[\nu](\pi_c)$ has already been analysed in § (4.2). In particular

$$\Gamma_{[n]} = A^1, \tag{4.20}$$

$$\Gamma_{[1^n]} = \begin{cases} A^1, & n/2 \text{ even} \\ A^2, & n/2 \text{ odd.} \end{cases} \tag{4.21}$$

The remaining UIR are best symmetrized using the theory of § 3.3. Thus

$$(T_1)^{[v]} = (D^1)^{[v]} \downarrow O, \tag{4.22}$$

$$(T_2)^{[v]} = T_1^{[v]} \otimes A_2^{[n]}, \tag{4.23}$$

$$(\bar{E}_1)^{[v]} = (D^{1/2})^{[v]} \downarrow O', \tag{4.24}$$

$$(\bar{E}_2)^{[v]} = (\bar{E}_1)^{[v]} \otimes (A_2)^{[n]}, \tag{4.25}$$

$$(\bar{F})^{[v]} = (D^{3/2})^{[v]} \downarrow O'. \tag{4.26}$$

4.5. Real representations

In some applications it is necessary to work with representations that are irreducible over the real field rather than the complex field. We refer to Lyubarskii (1960) for a discussion of the importance of real representations to the theory of second-order phase transitions.

To symmetrize such representations, for instance ${}^1E\oplus{}^2E$ of $\boldsymbol{T}$, we would employ the formula

$$(L\oplus M)^{[v]} = \sum \sigma([v],[v'],[v''])(L^{[v']}\otimes M^{[v'']}), \tag{4.27}$$

where $[v]$, $[v']$, $[v'']$ are UIR of $\boldsymbol{S}_n$, $\boldsymbol{S}_{n'}$, $\boldsymbol{S}_{n''}$, respectively, where $n = n'+n''$, and σ is the frequency of $[v]$ in $([v']\otimes[v''])\uparrow \boldsymbol{S}_n$, induced from the subgroup $\boldsymbol{S}_{n'}\times\boldsymbol{S}_{n''}$. This is an easy consequence of the Weyl formula and is the basis for Gard and Backhouse (1974). Equation (4.27) is especially simple to employ if L, M are one dimensional, then σ takes the value 1 or 0, depending on whether the Young's diagram of $[v]$ may or may not be built from those of $[n']$ and $[n'']$.

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